AIAA Journal
Vol. 46, No. 6, June 2008

Robustly Optimal Fixed Pitch Wind Turbine with Tip Correction and Drag

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DOI: 10.2514/1.34055

The trisection of nominal wind angle into apparent wind and blade pitch angles to optimize the horizontal axis wind turbine blade element lift power and robustly maintain the optimum as the wind varies is corrected for drag and at the blade tips. The best compromise of robust vs minimum drag-to-lift design angle of attack is analyzed, and the robust is shown to dominate for the typical total variance of the nominal angle from the wind and load regimes and windshear. A happy confluence emerges in the outer blade of requirements for low drag to lift at small robust design angles of attack with prompt stall regulation.

Nomenclature

B	=	number of blades
C_D	=	sectional drag coefficient
C_P	=	rotor coefficient of power/undisturbed kinetic energy flux through the swept area
c	=	local blade chord
c_P	=	local elemental power coefficient per strip of swept area
e	=	lift slope correction factor for thickness
F	=	Prandtl tip correction function of f
f	=	nondimensional distance from the tip relative to the vortex sheet spacing s
g	=	exponent in the dependence of tip speed on wind speed
h	=	tip correction divisor for the robust angle of attack
$\boldsymbol{I}$	=	induction, half the velocity change the blades produce downstream in their wake
$\boldsymbol{J}$	=	induced flow at the blades
k	=	reduced frequency based on chord, $\Omega c/V$
$\boldsymbol{L}$	=	airfoil lift vector
m	=	factor for relative increase in wind speed with sine of azimuthal angle
$\boldsymbol{N}$	=	no-lift or nominal apparent wind
P	=	dimensional power per unit length of span
p	=	probability power density of wind
q	=	probability density of the wind
r	=	radius from the blade element to the axis of rotation
$\mathbf{r}$	=	unit radial vector
s	=	spacing between the tip vortex sheets
V	=	wind speed
$\boldsymbol{V}$	=	true undisturbed wind vector
V_d	=	design wind speed of peak wind power density
V_o	=	wind speed at the axis
W	=	magnitude of $\boldsymbol{W}$ and so on
$\boldsymbol{W}$	=	net apparent wind vector
X	=	the HAWT tip speed ratio
x	=	local speed ratio, $\Omega r/V$
$\mathbf{z}$	=	unit vector normal to the path directed downwind
α	=	3/4 chord midline angle of attack to $\boldsymbol{W}$
α_o	=	angle of attack of lowest drag-to-lift ratio
Δ	=	"a small numerical increment of..."
δ	=	"a small variation of..."
φ	=	true or complete apparent wind $\boldsymbol{W}$ angle to the blade path
λ	=	blade pitch or angle of the 3/4 blade chord to the blade path
ϕ	=	azimuthal angle from horizontal
ρ	=	fluid density
σ	=	true local solidarity, blade chords/circumference of blade travel $Bc/2\pi r$
τ	=	V/V_d
θ	=	nominal or no-lift apparent wind $\boldsymbol{N}$ angle to the blade path
Ω	=	angular velocity of rotation in radians per unit time
ϖ	=	half the net blade chord divided by the diameter, $e\pi\sigma/2 = eBc/4r$
$\swarrow$	=	tends down to from greater than (above)
$\nearrow$	=	tends up to from less than (below)

Subscript

d	=	denotes design value
m	=	denotes mean value over the wind spectrum
o	=	optimum value at $\varphi = 2\theta/3$

Introduction

A RECENT paper [1] solved exactly Glauert's [2] optimal horizontal axis wind turbine (HAWT) in blade element momentum (BEM) theory (without drag or the tip correction) and next found the blade element pitch and chord that made that optimum robust to small changes in the wind. Then the actual lift power coefficient operates only quartically below the ideal lift power around the design point to give a very broad optimum and benign avoidance of high load coefficients far from it. These solutions simply trisect the nominal wind angle θ. Two thirds is the optimal actual apparent wind angle φ and one third is the robust optimal blade pitch λ_r and the angle of attack α_r of the midline at the 3/4 chord point.

Stewart [3] had the related idea [1] of finding the pitch that optimized the rotor at two different operating tip speed ratios X, perhaps in analogy to the takeoff and cruise dual design of propellers. But most authors have followed Glauert [2] in pitching the wind rotor blade at its lowest drag-to-lift angle α_o to minimize the profile drag correction right at a single design wind speed for the highest net power there. A motive of this paper is to show whether this maximum height or the robust maximum breadth of the performance peak is the stronger influence on the net power in a real fluctuating wind. In fact, a linear equation to optimize the pitch for the best net power considering both will be developed.

This extended analysis will heavily use the nature of stationary points. For instance, the drag-to-lift ratio must increase with the square of the deviations of the angle of attack from α_o. Likewise, the

Received 19 August 2007; accepted for publication 24 November 2007.

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quadratic loss from the optimal lift power coefficient with wind variations will itself be proportional to the square of the deviation of the design angle α_d from α_r. Then, crudely, the optimal design α_d is just a weighted average of α_r and α_0, with the weights being their mean quadratic coefficients in the average power.

A careful analysis should begin by optimizing the net lift minus drag power at the design point in an extension of the new trigonometric formulations to include the drag angle ε [1].

BEM Optimum with Drag and Tip Correction

As in Fig. 1, consider an infinitesimal actuator area dA with solidity σ of a HAWT rotor with unit vectors radial $\boldsymbol{r}$, azimuthal $\boldsymbol{\phi}$, and $\boldsymbol{z}$ axial down the wind $\boldsymbol{V}$. Call the actual apparent wind $\boldsymbol{W}$, at angle φ to the rotor plane with $\boldsymbol{W} = \boldsymbol{N} + \boldsymbol{J}$, where $\boldsymbol{J}$ is the induced flow at the blades, $\boldsymbol{N}$ is the no-lift nominal wind at Glauert's angle $\theta(x) = \cot^{-1} x$ to the rotor plane, and x, the local speed ratio $\Omega r/V$. Then oppose the Joukowski airfoil lift d$\boldsymbol{L}$ by the net blade element area σdA, to the rate of change of momentum of the flux $\boldsymbol{W} \cdot \boldsymbol{z}dA$ of fluid density ρ passing through dA due to its net downstream velocity change 2$\boldsymbol{I}$ due to the lift.

The BEM equation:

$$\tfrac{1}{2}\rho W C_L \mathbf{r} \mathbf{x} \boldsymbol{W} \sigma = -\mathrm{d}\boldsymbol{L}/\mathrm{d}A = 2\rho \boldsymbol{I}(\boldsymbol{W} \cdot z) \tag{1}$$

ignoring any net pressure force on the stream tube walls and ends. The elemental profile drag $\mathrm{d}\boldsymbol{D} = \frac{1}{2}\rho W C_D \boldsymbol{W} \sigma \mathrm{dA}$ is perpendicular to d$\boldsymbol{L}$. Therefore, the net force is a rotation of d$\boldsymbol{L}$ through the drag-to-lift angle $\varepsilon = \tan^{-1} C_D/C_L$. This drag arises by flow retardation in the boundary layer, and so it does not affect $\boldsymbol{J}$, the outer inviscid flow at the B blades induced by the lift.

Prandtl [3] considered this induced flow around the vortex sheets trailing from the B blades at relative velocity $\boldsymbol{W}$, angled at φ, and spaced at $s = 2\pi \sin\varphi r/B \approx$ advance$/B$. These move downstream relative to the wind $\boldsymbol{V}$ at self-induced 2$\boldsymbol{J}$ perpendicular to $\boldsymbol{W}$. He found $\boldsymbol{J} = \mathbf{I}/F$, F a function of $(R - r)/s$ as the induced flow at the blades, and showed this reduced the effective length of the blade by $.221s$. His implicit blade circulation $cC_L W/2 = 2JsF$ follows F from zero at the tip to unity inboard.

Inboard and considering the axisymmetry, $\boldsymbol{J}$ also tends to $\boldsymbol{I}$ in the standard BEM model of downstream dissipation of the entire swirl kinetic energy added at the rotor [4]. If the swirl can expand before dissipating, the BEM is conservative and modified momentum equations have been solved, partly numerically, to give the optimum blade elements in the middle of the blade, which are only slightly different [4].

With this $\mathbf{I} = \boldsymbol{J}/F$ closure, the net power P can be expressed in terms of the angles of Fig. 1 as

$$\mathrm{d}P = xV\boldsymbol{\phi} \cdot (d\boldsymbol{L} + d\boldsymbol{D}) = 2\rho x V(\boldsymbol{W} \cdot z)\mathrm{I}(\sin\varphi - \cos\varphi \tan\varepsilon)\mathrm{d}A \tag{2}$$

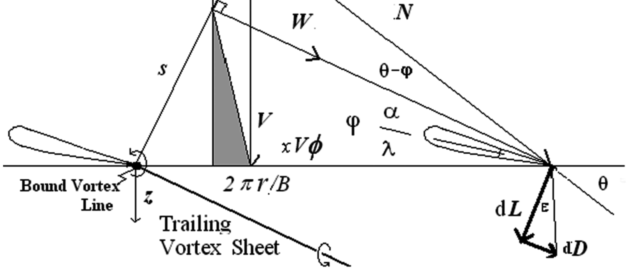

Fig. 1 Velocity and vortices at the blades.

$$J = N\sin(\theta - \varphi), \qquad \boldsymbol{W} \cdot \boldsymbol{z} = W\sin\varphi, \qquad W = N\cos(\theta - \varphi)$$
$$\mathrm{d}P/\mathrm{d}A = \rho x V N^2 F \sin 2(\theta - \varphi)\left(\sin^2\varphi - \tfrac{1}{2}\sin 2\varphi \tan\varepsilon\right) \tag{3}$$

To optimize for a set, r, R, and$V$ so x, ξ, and F fixed by set Φ, the tip φ; zero the φ derivative of dP/dA

$$\rho x V F N^2 (2\sin\varphi \sin(2\theta - 3\varphi) - \sin(2\theta - 4\varphi)\tan\varepsilon) = 0 \tag{4}$$

Expanding $2\theta - 4\varphi$ as $(2\theta - 3\varphi) - \varphi$ gives

$$2\sin\varphi \sin(2\theta - 3\varphi)\cot\varepsilon = \cos\varphi \sin(2\theta - 3\varphi) - \sin\varphi \cos(2\theta - 3\varphi) \tag{5}$$

so

$$2\cot\varepsilon = \cot\varphi + \cot(3\varphi - 2\theta) \tag{6}$$

$\varphi = 0$ is near a minimum and $3\varphi = 2\theta$ near a maximum. A second approximation to the latter is

$$\varphi_o = \tfrac{2}{3}\theta + \varepsilon/6 + O(\varepsilon^2) \quad \text{or} \quad \theta = 3\varphi_o/2 - \varepsilon/4 \tag{7}$$

The drag effect is very weak given that typical ε are less than a degree. Then the optimum angular interference $1 - \varphi/\theta$ of the induced flow $\boldsymbol{J}$ at the blades virtually remains 1/3 at all x and all F, and all the new results of Section 3 in Farthing [1] still hold with the interferences defined relative to J, and the power factored by F and drag corrected. With $c_{\mathrm{p}} = \mathrm{d}P/\mathrm{d}A/\frac{1}{2}\rho V^3$

$$c_{po} = 2x(1 + x^2)F\sin(\varphi_o - \varepsilon/2)\left(\sin^2\varphi_0 - \tfrac{1}{2}\varepsilon \sin 2\varphi_o\right) \tag{8}$$

With Eq. (4), $\mathrm{d}^2 c_p/\mathrm{d}^2\varphi$ is

$$2x(1 + x^2)F(-6\sin\varphi + \varepsilon[4\cos(2\xi - 4\varphi) - \cos\varphi]) \approx 2x(1 + x^2)F(-6\sin\varphi_o + 3\varepsilon\cos\varphi_o) \tag{9}$$

Also, the variation of the circulation can be checked vs the tip flow model. From Eq. (1), one gets $\frac{1}{2}W^2 c C_L = 2FJWs$, and so the circulation $\frac{1}{2}WcC_L = 2FJs$, as implicit in Prandtl. Attempts to refine the average momentum flux in Eq. (1) due to the Prandtl variation lose this consistency and will not be pursued here.

Linear Lift Robustly Optimal Blade Elements

The BEM Eq. (1) and Fig. 1 give

$$\sigma C_L = 4F\sin\varphi \tan(\theta - \varphi) \tag{10}$$

At modest 3/4 chord angle of attack α, $C_L/2\pi = e\sin\alpha$, where e is a thickness factor slightly decreasing the theoretical lift slope. Because Eq. (7) gives optimally

$$\theta - \varphi_o = \tfrac{1}{2}\varphi_o - \tfrac{1}{4}\varepsilon \tag{11}$$

then

$$\varpi \sin\alpha = F\sin\varphi \tan\left(\tfrac{1}{2}\varphi_o - \tfrac{1}{4}\varepsilon\right) \tag{12}$$

if $\varpi = \frac{1}{2}e\pi\sigma = \frac{1}{4}eBc/r$ for B blades each of chord c.

A new way to optimally resolve Glauert's "indeterminacy" [1] between ϖ or σ, and α or C_L at the design θ_{d}, is to require a specific physical blade element fixed at 3/4 chord pitch angle $\lambda(r) = \varphi_o - \alpha$ and chord $c(r)$ to "robustly" follow the optimal for a small change in $\theta(x)$ about θ_{d}, due to a variation in the wind. By differentiation in φ_o, $\varpi \sin(\varphi_o - \lambda_r) = F\sin\varphi \tan(\frac{1}{2}\varphi_o - \frac{1}{4}\varepsilon)$ despite φ_o varying with θ about φ_d at set ϖ and λ requires

$$\varpi\cos(\varphi_o-\lambda_r)=F\cos\varphi_o\tan\left(\tfrac{1}{2}\varphi_o-\tfrac{1}{4}\varepsilon\right)+\left(\tfrac{1}{2}-\tfrac{1}{4}\mathrm{d}\varepsilon/\mathrm{d}\alpha\right)F\sin\varphi_o\sec^2\left(\tfrac{1}{2}\varphi_o-\tfrac{1}{4}\varepsilon\right)+\sin\varphi_o\tan\left(\tfrac{1}{2}\varphi_o-\tfrac{1}{4}\varepsilon\right)dF/d\varphi_o \quad (13)$$

because the rotor tip φ, Φ will change too. Dividing Eqs. (13) by (12) at $\varphi_d=\varphi_o$ gives

$$\cot\alpha_r=\cot\varphi_d+\left(1-\tfrac{1}{2}\mathrm{d}\varepsilon/\mathrm{d}\alpha\right)\csc\left(\varphi_d-\tfrac{1}{2}\varepsilon\right)+\mathrm{d}F/F\mathrm{d}\varphi_o \quad (14)$$

Taylor expanding the second term gives

$$\cot\alpha_r\approx\cot\tfrac{1}{2}\varphi_d+\tfrac{1}{2}\varepsilon\csc\varphi_d\cot\varphi_d-\tfrac{1}{2}\mathrm{d}\varepsilon/\mathrm{d}\alpha\csc(\varphi_d)+\mathrm{d}F/F\mathrm{d}\varphi_o \quad (15)$$

Well inside of the tip $\mathrm{d}F/\mathrm{d}\varphi_o=0$ and then Taylor expanding $\cot^{-1}$ of the right side gives

$$\alpha_r\approx\frac{1}{2}\varphi_d-\varepsilon\left(1-\tan^2\tfrac{1}{2}\varphi_d\right)\Big/8+\tfrac{1}{4}\mathrm{d}\varepsilon/\mathrm{d}\alpha\tan\left(\tfrac{1}{2}\varphi_d\right)+O(\varepsilon^2) \quad (16)$$

Therefore, $\alpha_r\approx\frac{1}{3}\theta-\varepsilon/24$, showing drag to little perturb the trisection, just as it had little effect on Stewart's dual optimum [3]. The $dF/Fd\varphi_o$ term is simplest to evaluate by taking $s=2\pi\sin\varphi_o r/B$ as commonly used. Now $\cos(\pi F/2)=e^{-f}$, and so $\pi\mathrm{d}F=2\cot(\frac{1}{2}\pi F)\mathrm{d}f \quad f=\pi(R-r)/s=B(R-r)/2r\sin\varphi_o$ and so $\mathrm{d}f=-f\cot\varphi_d\mathrm{d}\varphi_o$

$$\cot\alpha_r=\cot\varphi_d+\left(1-\tfrac{1}{2}\mathrm{d}\varepsilon/\mathrm{d}\alpha\right)\csc\left(\varphi_d-\tfrac{1}{2}\varepsilon\right)-2f\cot\left(\tfrac{1}{2}\pi F\right)\cot\varphi_d/\pi F \quad (17)$$

Using small angle approximations for α_r and φ_d appropriate at the tip's large X

$$\alpha_r\approx\tfrac{1}{2}\varphi_d/h,\qquad h=1+\tfrac{1}{4}\varepsilon/\varphi_d-\tfrac{1}{4}\mathrm{d}\varepsilon/\mathrm{d}\alpha-f\cot\left(\tfrac{1}{2}\pi F\right)\Big/\pi F \quad (18)$$

At the very tip in the limit of small f, $2f=(\frac{1}{2}\pi F)^2$ so that the last term in Eq. (18) tends to $-\frac{1}{4}$, reflecting the $\frac{1}{2}$ exponent of potential flow around an edge [5]. Then $h\approx 3/4+\frac{1}{4}\varepsilon/\varphi_d+O(\varepsilon^2)$. Throughout, the robust ϖ_r is given by

$$F\sin\varphi_d\tan\left(\tfrac{1}{2}\varphi_d-\tfrac{1}{4}\varepsilon\right)\Big/\sin\alpha_r \quad (19)$$

Therefore, the robust blade reduces its chord by $\frac{3}{4}F$ very near the tip from the constant chord beyond the tip region (when the drag correction is ignored).

Let $\Delta\lambda=\lambda-\lambda_{r=}\alpha_r-\alpha_d$ and $\Delta\varpi$ be small deviations from robust values. Then, for the same $\varpi\sin(\varphi-\lambda)$ at $\varphi=\varphi_o$, one must have $\Delta\varpi=\Delta\lambda\varpi_r\cot\alpha_r$. Differentiating the nonoptimal linear lift BEM

$$\varpi\sin(\varphi-\lambda)=F\sin\varphi\tan(\xi-\varphi) \quad (20)$$

$$\mathrm{d}\varphi/\mathrm{d}\theta=F\sin\varphi\sec^2(\theta-\varphi)/\{F\sin\varphi\sec^2(\theta-\varphi)-F\cos\varphi\tan(\xi-\varphi)+\mathrm{d}F/\mathrm{d}\varphi\sin\varphi\tan(\theta-\varphi)+\varpi\cos\alpha\} \quad (21)$$

Call this u/v to be evaluated at the central design point $\varphi=\varphi_o$. Now, if ϖ and λ are robust, then it must be $\mathrm{d}\varphi_o(\theta)/\mathrm{d}\theta$, so that any variation must be due to nonrobust $\varpi\cos\alpha$ in the denominator. The robust denominator v in Eq. (22) must and does simplify eventually, but the exercise can be avoided using the known robust values of u/v and u

$$\mathrm{d}(\varphi-\varphi_o)/\mathrm{d}\theta\approx-(\mathrm{d}\varphi_o/\mathrm{d}\theta)^2\Delta(\varpi\cos\alpha)/u \quad \text{now } \mathrm{d}\varphi_o/\mathrm{d}\theta=2/3 \quad (22)$$

$$\Delta(\varpi\cos\alpha)=\varpi\Delta\lambda/\sin\alpha \quad\text{and}\quad u=F\sin\varphi\sec^2(\theta-\varphi)$$

$$\mathrm{d}(\varphi-\varphi_o)/\mathrm{d}\theta\approx-4/9\Delta\lambda\tan(\theta-\varphi)\cos^2(\theta-\varphi)/\sin^2\alpha_r\approx-4\Delta\lambda h^2/9\tan\left(\tfrac{1}{2}\varphi_d\right) \quad (23)$$

upon dropping the small drag correction in the factors, because the global optimum Δs will prove next to be very small.

Outer Blade Element Optimum Blade Pitch with Drag

The net power at the design point is lowest for the smallest C_D/C_L ratio ε_o, which occurs at the best α_o. But any variation of the wind will perturb the independent variable θ and so the operating point $\varphi(\theta)$ away from the design. Then, to maintain the optimum $\varphi_o(\theta)$, the design angle of attack should be the robust α_r found above. This section finds the best compromise α_d between c_p peak breadth in α_r and height in α_o for the highest mean power.

For a given θ, the net c_p is stationary at the optimum $\varphi_o(\theta)$ (and ε_o), and from Eq. (9) varies with φ as

$$c_p=c_{po}(\theta)-x(1+x^2)F(6\sin\varphi_o-3\varepsilon_o\cos\varphi_o)(\varphi-\varphi_o)^2 \quad (24)$$

where $(\varphi-\varphi_o)$ will be Taylor expanded in terms of the $\delta\theta$ variation of θ from Eq. (23). This explicitly shows how the deviation of the power from the robust always optimal varies as the square of not only $\delta\theta$ [1] but also the square of the deviation of the pitch from the robust value. If the pitch is robust, the next even order quartic $(\delta\theta)^4$ term must lead. If ε is simultaneously varied from ε_o, add $(\varepsilon-\varepsilon_o)\{1+(\varphi-\varphi_o)\mathrm{d}/\mathrm{d}\varphi\}\mathrm{d}c_p/\mathrm{d}\varepsilon$ for the increase in the drag loss at constant θ or

$$x(1+x^2)F\left\{-\tfrac{1}{2}\sin\varphi_d\sin2\varphi_d+\sin\left(\varphi_d+\tfrac{1}{2}\varepsilon_o\right)(\varphi-\varphi_o)\right\}\times\mathrm{d}^2\varepsilon/\mathrm{d}\alpha^2(\alpha-\alpha_o)^2 \quad (25)$$

For the NACA 23018 [6], $e=0.91$, $\alpha_o=0.175=10$ deg, $\varepsilon_o=1/80=0.7$ deg, and $b=\mathrm{d}^2\varepsilon/\mathrm{d}\alpha^2\approx0.45$. This predicts a 61% higher $\varepsilon=1.13$ at $\alpha=0$ or 20 deg, whereas ε is really infinite at $\alpha=0$ and high at stalled 20 deg, so that its range is less than say ±5 deg without higher order terms. Of course, the airfoil shape should be tuned along the wing to keep α_o as close to α_r as possible. The method here finds the best compromise of the outstanding difference. Now $\alpha-\alpha_o$ can be expanded as $\alpha_d-\alpha_o+\varphi-\varphi_d$ because the pitch λ is fixed.

Averaging (denoted by underlining) over the small variation of φ equally above and below φ_d, and again evaluating trigonometric factors at the design point and ignoring small drag terms, the extra c_p variation from always optimal $c_{\mathrm{po}}(\theta)$ is approximately

$$x(1+x^2)Fb\sin\varphi_d\left[-\tfrac{1}{2}\sin2\varphi_d\{(\alpha_d-\alpha_o)^2+\underline{(\varphi-\varphi_d)^2}\}+2\underline{(\varphi-\varphi_o)(\varphi-\varphi_d)}(\alpha_d-\alpha_o)\right] \quad (26)$$

In the first expanding $\varphi-\varphi_d$ as $\varphi-\varphi_o+\varphi_o-\varphi_d$ and squaring generates a term in $(\varphi-\varphi_o)^2$, which is negligible vs the last term in Eq. (26) by $b\sin(2\varphi_d)/12$. In the second, the $(\varphi-\varphi_o)^2$ term is smaller by $b(\alpha_d-\alpha_o)/3$ or at least 0.026 as $|\alpha_d-\alpha_o|$ is less than α_o. Both have a cross factor of $\varphi-\varphi_o$ by $2(\varphi_o-\varphi_d)=4\delta\xi/3$. Thus, the α_d sensitive part of the net of the means of Eqs. (24) and (26) is

$$x_d\left(1+x_d^2\right)F\sin\varphi_d \times 6\underline{(\varphi-\varphi_o)^2} + \tfrac{1}{2}b\sin 2\varphi_d\{(\alpha_d-\alpha_o)^2 + 4\underline{(\varphi-\varphi_o)\delta\theta}/3\} - 8b\underline{(\varphi-\varphi_o)\delta\theta}/3(\alpha_d-\alpha_o) \tag{27}$$

And using $\varphi-\varphi_o=\delta\theta \mathrm{d}(\varphi-\varphi_o)/\mathrm{d}\theta$ and Eq. (23), this sensitive factor is Eq. (28):

$$6\underline{\left\{4(\alpha_{\underline{d}}-\alpha_{\underline{r}})\delta\theta\cot\tfrac{1}{2}\varphi h^2/9\right\}^2} + \tfrac{1}{2}b\sin 2\varphi_d\left\{(\alpha_d-\alpha_o)^2+16(\alpha_d-\alpha_r)\cot\tfrac{1}{2}\varphi\underline{(\delta\theta)^2}h^2/27\right\} - 32b(\alpha_d-\alpha_r)\cot\tfrac{1}{2}\varphi\underline{(\delta\theta)^2}(\alpha_d-\alpha_o)h^2/27 \tag{28}$$

Hence, the optimum α_d satisfies the linear equation (29):

$$32(\alpha_d-\alpha_r)\left[2h^4\cot^2\tfrac{1}{2}\varphi - bh^2\cot\tfrac{1}{2}\varphi\right]\underline{(\delta\theta)^2}/27 + (\alpha_d-\alpha_o)b\left[\sin 2\varphi_d - 32h^2\cot\tfrac{1}{2}\varphi\underline{(\delta\theta)^2}/27\right] = -8bh^2\sin 2\varphi_d\cot\tfrac{1}{2}\varphi\underline{(\delta\theta)^2}/27 \tag{29}$$

The last term in the first and second coefficients is at least 0.13 less than the first term in the first, so that the former is negligible and the latter confirms that for dominant $\underline{(\delta\theta)^2}$ variation α_d is much closer to α_r than to α_o.

As an example, consider the temporal variation of X and so all the $x=\cot\underline{\theta}$ with wind speed V around V_d with near constant rotor $C_P=XC_T$. For rapid fluctuations, rotor inertia stops the revolutions per minute (rpm) changing quickly. For a synchronous generator, the grid frequency keeps the rpm constant so that then X always varies as V^{-1}. For slow wind variations and an ideal cubic rpm load such as a fluid dynamic churn for mixing or heat, X is constant. For a permanent magnet generator connected to a dissipative load, the power consumed is proportional to rpm squared, so X varies as $V^{1/2}$. A positive displacement rotary pump against a fixed pressure head for unvarying load torque has V^2C_T constant or X varying as V^2. Therefore, consider in general $x\propto V^g$, $g^2>0$ even for cubic loads. Then, with $\tau=V/V_d$, $\delta\theta\approx\frac{1}{2}g\sin 2\theta\delta\tau/\tau_d$.

Whilst still ignoring the variation of trigonometric factors with θ varying from θ_d, here the time average will effectively include the V^3 scaling of the power available by using, with $q(\tau)$ the frequency of winds between τ and $\tau+\mathrm{d}t$, a power weighting $p(\tau)=\tau^3q(\tau)/\int\tau^3q(\tau)\,\mathrm{d}\tau$. This peaks at the peak power spectrum design point $\tau=1$ $V=V_d$, as the area (net power) under the multiplication of the two (parabolic) peaked curves is maximum when the peaks coincide. (Their abscissas θ and V are close to globally matched in the prime synchronous and wind shear cases of constant Ω where at large x, $\theta\propto V$.)

For the fairly tight Rayleigh distribution $q(\tau)=4\tau\exp(-2\tau^2)$ with mean V of $0.627V_d$ and a mean power density of 0.47 of $\frac{1}{2}\rho V_d^3$, matching the peak and its second derivative at V_d gives a consistent and underapproximation of quadratic about the peak $p(\tau)=3\{1-4(\tau-1)^2\}/2$ between zeros at $\tau=0.5$ and 1.5. Then, the $(\delta\tau)^2p(\tau)$ terms integrate to 1/20, so that dividing by $(\frac{1}{2}g\sin 2\theta)^2$ and multiplying by 270/4

$$(\alpha_d-\alpha_r)8h^2\cot^2\tfrac{1}{2}\varphi_d + (\alpha_d-\alpha_o)b\left[270\sin 2\varphi_d/g^2\sin^2 3\varphi_d - 4h^2\cot\tfrac{1}{2}\varphi_d\right] \approx -bh^2\sin 2\varphi_d\cot\tfrac{1}{2}\varphi_d \tag{30}$$

Figure 2 presents $g^2=1$ calculations for the NACA 23018 lift-to-drag data above. The optimal local coefficient of performance $c_P(x)$ was calculated from Eq. (8) and then numerically integrated to get the rotor coefficient $C_{Po}=\int c_{Pm}\,\mathrm{d}x^2/X^2$, which peaks when $C_{Po}=c_{Po}$. The intersection of these curves helps to locate the peak C_{Po} at $X=5$ $\theta=11.3$ deg, as its maximum is very broad. To give this effective tip speed ratio, the actual tip speed ratio must be increased relatively by $0.221\ 2\pi\sin\varphi/B$ or, for the standard $B=3$, by 1.062 to 5.31.

The constant term in Eq. (30) affects the results by less than 3%, and so, ignoring that, the best α_d is indeed a weighted average of α_r and α_o as anticipated in the Introduction. Iterating in α_r Eqs. (16) and (18) to correct for its ε being greater than ε_o and for its $\mathrm{d}\varepsilon/\mathrm{d}\alpha$ had no noticeable effect. Actually α_d does not exceed the stall value due to the moderating influence of the relatively small α_o at small x. The leading coefficients are equal at about $\varphi=30$ deg or $x=1$, sothat the robust α_r increasingly dominates the minimum drag α_o moving out in the robust zone. Conversely, the sum of the two quadratic losses from the ideal highest and always optimal is in turn dominated by the reduction of the peak height by the nonminimal drag/lift shown as $c_{p\varepsilon}$ vs $c_{p\varepsilon o}$.

The lines above the tip X show the design angle of attack and chord with no tip. The best angle of attack then follows above the robust at a proportion $(\alpha_d-\alpha_r)/\alpha_r$ rising to $3.7b\alpha_o/g^2$ or 29% above. The nondimensional net design chord, *Bek*, slowly declines because of the rise both in this proportion and in the drag correction factor in the robust chord, agreeing with Stewart [3]. Without these drag effects the design = robust *Bek* rose to an asymptote of 8/3 and, of course, c_p and Cp rise to an asymptote of $16/27=0.593$ [1].

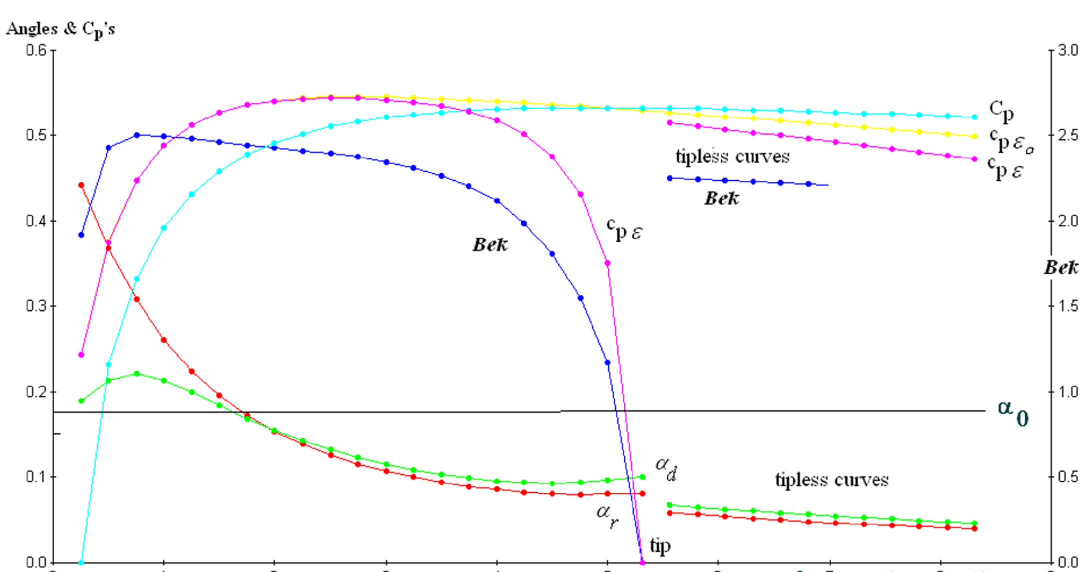

Fig. 2 Optimum HAWT blade design chord and angle of attack.

The lines just below the tip X show the Prandtl F influence in raising the robust angle of attack and the design proportion above it due to the local weakening of the robust dominance with h^4, so much so that the design angle reaches a minimum of 0.096 or 5.5 deg at about $x = 4.5$. Synchronously, at $g = -1$, these help to relieve the high wind tip bending moment by reducing its design stall margin.

Really, $\underline{(\delta\theta)^2}$ and $\underline{(\delta\tau)^2}$ could be generalized as variances about the design point. To do so for the open-ended Rayleigh distribution brings in too much variation far from the peak and raises the above close-to-peak value of 0.05 by an inappropriate amount to 0.118. But statistical formulas are suitable for the tighter fixed end distribution generated by the blade element sweeping through windshear.

Thus, if the wind shears linearly as $V_0(1 + m \sin\phi)$, where $m = kx/X$ to be $V_0(1 + k)$ at the tip's highest point azimuth $\phi = \pi/2$, the cubic power weighting is $p(\phi) = (1 + m \sin\phi)^3/a$, where the integration will be averaging over ϕ, and so then $a = 1 + 3m^2/2$. Then the variation $\delta\tau_0 = m \sin\phi$ in $\tau_0 = V/V_0$ has power weighted averages $a\underline{\delta\tau_0} = 3m^2/2 + 3m^4/8$ and $a\underline{(\delta\tau_0)^2} = \frac{1}{2}m^2 + 9m^4/8$. The power mean $\underline{\tau}_0$ of $1 + \underline{\delta\tau_0} \approx 1 + 3m^2/2 - 15m^4/8$ and the true minimum variance about it is $\underline{(\delta\tau_0)^2} - \underline{\{\delta\tau_0\}} \approx \frac{1}{2}m^2 - 15m^4/8$, so that, finally taking the design as this power mean, $\underline{\{\delta\tau\}^2}$ is about $\frac{1}{2}m^2 - 27m^4/8$. So at $k = 1/3$ when the top wind is twice the bottom and at representative $x/X = 0.7$ $\underline{\{\delta\tau\}^2}$ is 0.017 vs 0.05 Rayleigh, and so the robust would then be about one third as dominant for an ideal cubic load as calculated above. Calculated exactly for the tip, $\{\underline{\delta\tau}\}^2$ is 0.029. When $g > 0$, this independent variance must be added to the Rayleigh-derived one, and its increase toward the tip helps to make the robust more dominant and more uniformly so.

Every other independent source of variation likewise adds further to the dominance of the robust blade setting. The principal unaccounted (hard to quantify) variation is in the wind direction, potentially very significant for big HAWTs with their active yaw power consumption (but in yaw the trisection has only been proved in the limit of high x with θ reduced as the cosine of the yaw [1]). Then, the θ variance would be proportional to the mean of the fourth power of the yaw angle, which would need to be averaged from wind data and the HAWT yaw response.

Conclusions

The drag and tip-corrected blade element momentum theory that is the basis of most HAWT computer design has been analytically optimized for maximum annual power. These refinements do not substantially affect the optimum apparent wind angle φ being 2/3 of the nominal θ. But the design angle of attack to robustly maintain the optimum as the wind varies is substantially increased above the naive $\frac{1}{3}\theta$ in the tip region because the tip loss (1-F) increases with wind speed at fixed rpm.

The section should be chosen with the minimum drag-to-lift ratio at the robust angle of attack. This favors increasingly thin sections on the outer blade. If the angle of attack for its best drag to lift is different from the robust, a linear equation gives the compromise design angle of attack with the best mean annual power in terms of the wind and load regimes. The typical domination of the robust angle outside the blade root is moderated by the tip correction, again raising the design angle of attack in the tip zone, which along with the optimal thin sections helps its stall regulation of bending moment.

Thus, this paper has furthered the goal of its predecessors [1,4] in revealing further blessed technical confluences that have helped make fixed pitch HAWTs such a practical renewable energy success.

Acknowledgment

Thanks to Robin East of Southampton for sacrificing some of his retirement to encourage and check this analysis and to the referees.

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